Licenciatura em Gestão

Operational Research Chapter 2

2017-2018

100 ANOS A PENSAR NO FUTURO

2. Simplex Method

Prototype Example 1

x_1 – no. batches of P1 produced per week (P1=8-foot glass door with aluminum framing)

x_2 – no. batches of P2 produced per week (P2=4×6 foot double-hung wood framed window)

Z – total profit per week (in thousands of dollars) from producing these two products

Linear Programming (LP) Model:

$$Max\ Z = 3x_1 + 5x_2$$

$$s.t.\begin{cases} x_1 & & \leq 4 \\ & 2x_2 & \leq 12 \\ 3x_1 + & 2x_2 & \leq 18 \\ & x_1, x_2 \geq 0 & \end{cases}$$

Linear Programming

Definitions I (Recall)

Solution of an LP – a vector of R^n which components are the values of the variables;

Feasible Solution (FS) – a solution that satisfies all the constraints (functional and sign);

Non Feasible Solution (NFS) – a solution that does not satisfy at least one of the constraints;

Feasible Region (FR) – the set of all feasible solutions;

Optimal Solution (OS) – a feasible solution that gives the best value to the objective function (OF) (the best value=maximum or minimum);

Optimal value – the value of the objective function at an optimal solution;

Binding constraint in a solution – a constraint that hold with equality at that solution;

To solve an LP is to determine the optimal solution (or solutions) and the optimal value or to conclude that an optimal solution does not exist and why.

Properties (Recall)

Prop 1: The Feasible Region of an LP problem is either an empty set or a convex set.

Prop 2: If the Feasible Region of an LP problem is nonempty and bounded then at least an optimal solution exists.

Prop 3: If an LP problem has optimum then at least one of its corner point feasible solutions (CPF) is an optimal solution.

Prop 4: Given an LP Problem with optimum, if a CPF has no adjacent CPF with a better value for the Objective Function then that point is an optimal solution.

LP – The Simplex Method

Prop 3: If an LP problem has optimum then at least one of its corner point feasible solutions (FCP) is an optimal solution. (Recall)

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 \leq 4 & (3) \\ 2x_2 \leq 12 & (4) \\ 3x_1 + 2x_2 \leq 18 & (5) \\ x_1 \geq 0 & (1) \\ x_2 \geq 0 & (2) \end{cases}$$

5 constraints 2 binding

↓

maximum $C_2^5 = \frac{5!}{2!(5-2)!}$ solutions !

$$\begin{matrix} (3) \\ (4) \end{matrix} \begin{cases} x_1 = 4 \\ 2x_2 = 12 \end{cases} \begin{cases} x_1 = 4 \\ x_2 = 6 \end{cases}$$

Feasible ?

$$\begin{cases} 3x_1 + 2x_2 \leq 18 & (5)\ \text{False} \\ x_1 \geq 0 & (1)\ \text{OK} \\ x_2 \geq 0 & (2)\ \text{OK} \end{cases}$$

NO!

Binding constraints

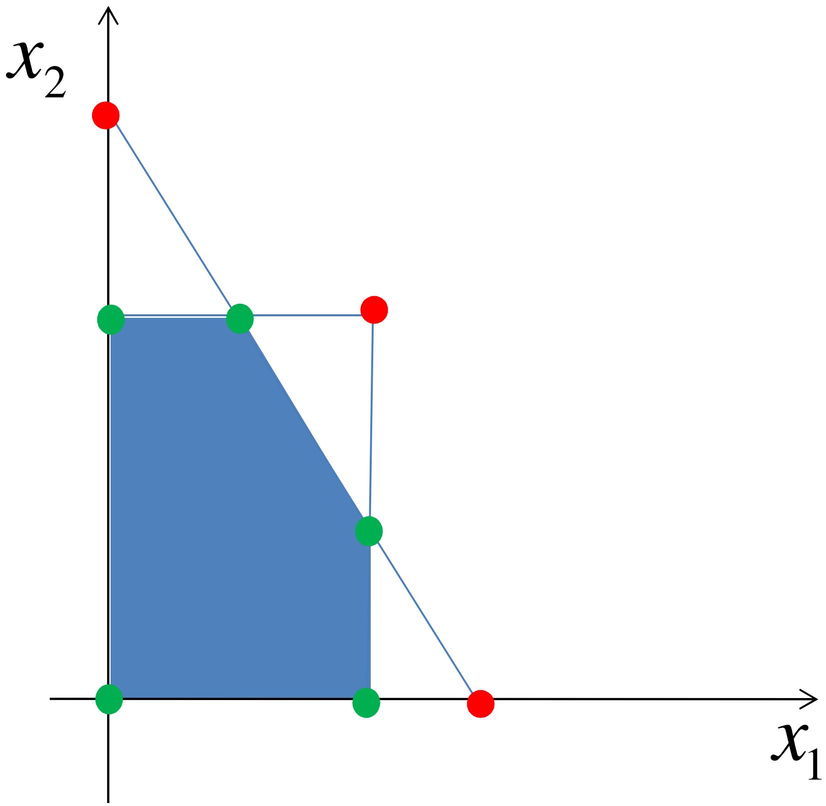
solution

Feasible / infeasible

$x_1 \ge 0$	$x_2 \ge 0$	$x_1 \le 4$	$2x_2 \le 12$	$3x_1 + 2x_2 \le 18$	x_1	x_2	
Yes	Yes				0	0	0
Yes		Yes			No sol.		
Yes			Yes		0	6	30
Yes				Yes	0	9	
	Yes	Yes			4	0	12
	Yes		Yes		No sol.		
	Yes			Yes	6	0	0
		Yes	Yes		4	6	
		Yes		Yes	4	3	27
			Yes	Yes	2	6	36

Evaluate the objective function value of the feasible solutions

Choose the best!



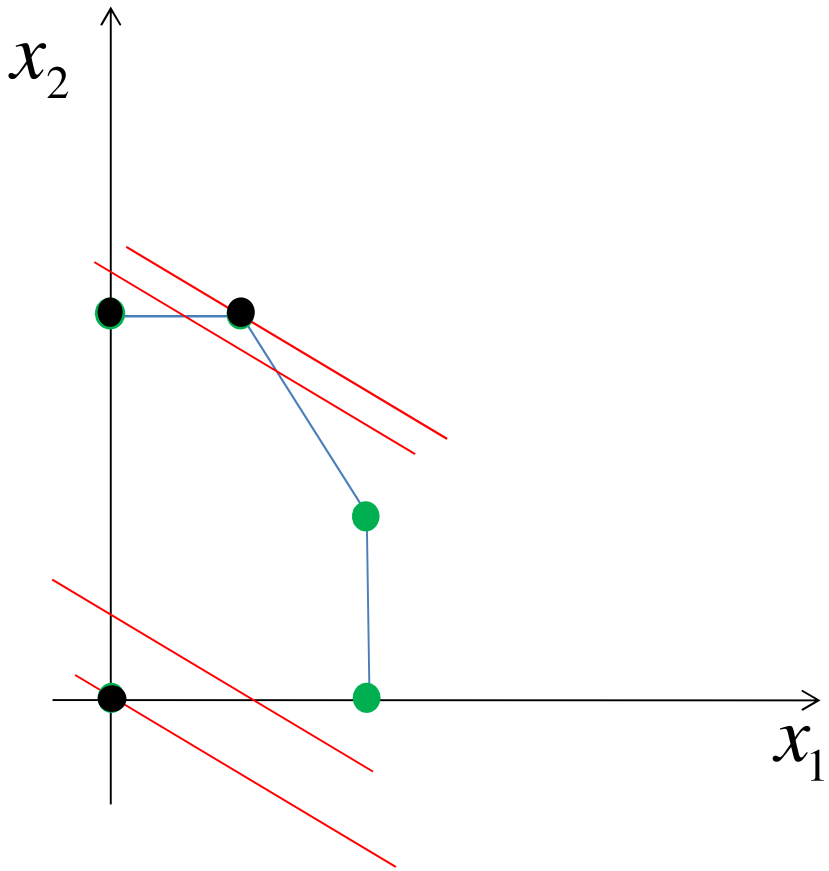
solution

Feasible / infeasible

x_1	x_2
0	0
No sol.	
0	6
0	9
4	0
No sol.	
6	0
4	6
4	3
2	6

LP – The Simplex Method

Prop 4: Given an LP Problem with optimum, if a FCP has no adjacent FCP with a better value for the OF then that point is an optimal solution.(Recall)



LP – The Simplex Method

In this course

Simplex method to solve LP problems in the standard form and nonnegative right-hand-sides .

Definitions

Standard form of an LP – a maximization problem
+ all functional constraints expressed by inequalities of the form ≤
+ all variables nonnegative (≥ 0);

Augmented form of an LP – LP model where
+ all variables nonnegative (≥ 0)
+ all functional constraints expressed by equations (using nonnegative variables - slack variables or surplus variables);

Augmented solution is a solution to the LP problem in the augmented form.

Properties

Prop 5: Every LP problem can be rewritten as an equivalent problem in the maximisation augmented form.

Writing an LP problem as a maximization augmented form problem

Objective: $\text{Min } Z = \sum_{j=1}^{n} c_j x_j \Leftrightarrow \text{Max}(-Z) = \sum_{j=1}^{n} (-c_j) x_j$ $\quad (\text{Min } Z = -\text{Max}(-Z))$

Constraints:

"≤": $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \wedge x_{n+i} \geq 0$

"≥ ": $\sum_{j=1}^{n} a_{ij} x_j \geq b_i \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i \wedge x_{n+i} \geq 0$

Variables:

$x_j \leq 0 \Leftrightarrow x_j = -x'_j \quad (x'_j \geq 0)$

$x_j \text{ livre} \Leftrightarrow x_j = x'_j - x''_j \quad \left(\begin{cases} x'_j = \text{Max}\{0; x_j\} \geq 0 \\ x''_j = \text{Max}\{0; -x_j\} \geq 0 \end{cases} \right)$

Definitions

Consider a problem with m equations and ℓ (>m) nonnegative variables.

Set to zero $\ell - m$ nonbasic variables (NBV). If possible, in a unique way, solve the system of linear equations to obtain the value of the remaining m variables – basic variables (BV)

→ Basic Solution (BS).

Basic Feasible Solution (BFS) – is a BS with all variables verifying the signal constraints. Otherwise the solution is Basic InFeasible Solution (BIFS).

1 BFS → 1 corner point of the FR (Feasible Region)

1 corner point of the FR → at least one BFS

Adjacent basic solutions - only one different BV (In $\mathbb{R}^2$ - extreme points of a line segment)

LP – The Simplex Method

Prototype augmented form

$$Max\ z = 3x_1 + 5x_2$$

$$s.t. \begin{cases} x_1 \leq 4 \\ 2x_2 \leq 12 \\ 3x_1 + 2x_2 \leq 18 \\ x_1,\ x_2 \geq 0 \end{cases}$$

$$Max\ z = 3x_1 + 5x_2$$

$$s.t. \begin{cases} x_1 + x_3 = 4 \\ 2x_2 + x_4 = 12 \\ 3x_1 + 2x_2 + x_5 = 18 \\ x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0 \end{cases}$$

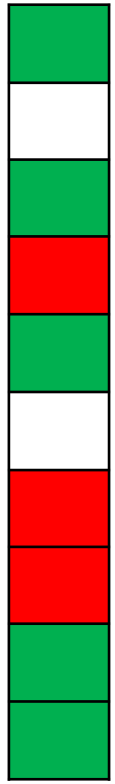
Binding constraints

x_1	x_2	x_3	x_4	x_5
0	0	4	12	18
0	-	0	-	-
0	6	4	0	6
0	9	4	-6	0
4	0	0	12	6
-	0	-	0	-
6	0	-2	12	0
4	6	0	0	-6
4	3	0	6	0
2	6	2	0	0

solution

Feasible / infeasible

x_1	x_2
0	0
No sol.	
0	6
0	9
4	0
No sol.	
6	0
4	6
4	3
2	6



LP –simplex algorithm

$$Max\ z = 3x_1 + 5x_2 = 0$$

$$s.t.\begin{cases} x_1 + x_3 = 4 & x_2 \leq \infty \\ 2x_2 + x_4 = 12 & x_2 \leq 6 \ (\frac{12}{2}) \\ 3x_1 + 2x_2 + x_5 = 18 & x_2 \leq 9 \ (\frac{18}{2}) \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

	Bas		coefficients							
iter	Var	Eq	Z	x_1	x_2	x_3	x_4	x_5	RHS	Op?
0	Z	0	1	-3	-5	0	0	0	0	NO
	x_3	1	0	1	0	1	0	0	4	
	x_4	2	0	0	2	0	1	0	12	
	x_5	3	0	3	2	0	0	1	18	

x_2 basic

$$Min\{\frac{12}{2}, \frac{18}{2}\}$$

x_4 non basic

LP –simplex algorithm (George Dantzig, 1947)

0. Consider an LP problem in the *standard* form with all RHS nonnegative.

Initialization

1. Write the problem in the *augmented* form
2. Determine an initial BFS: NBV = decision variables and solve the system;
   Write the simplex tableau; $k \longleftarrow 1$

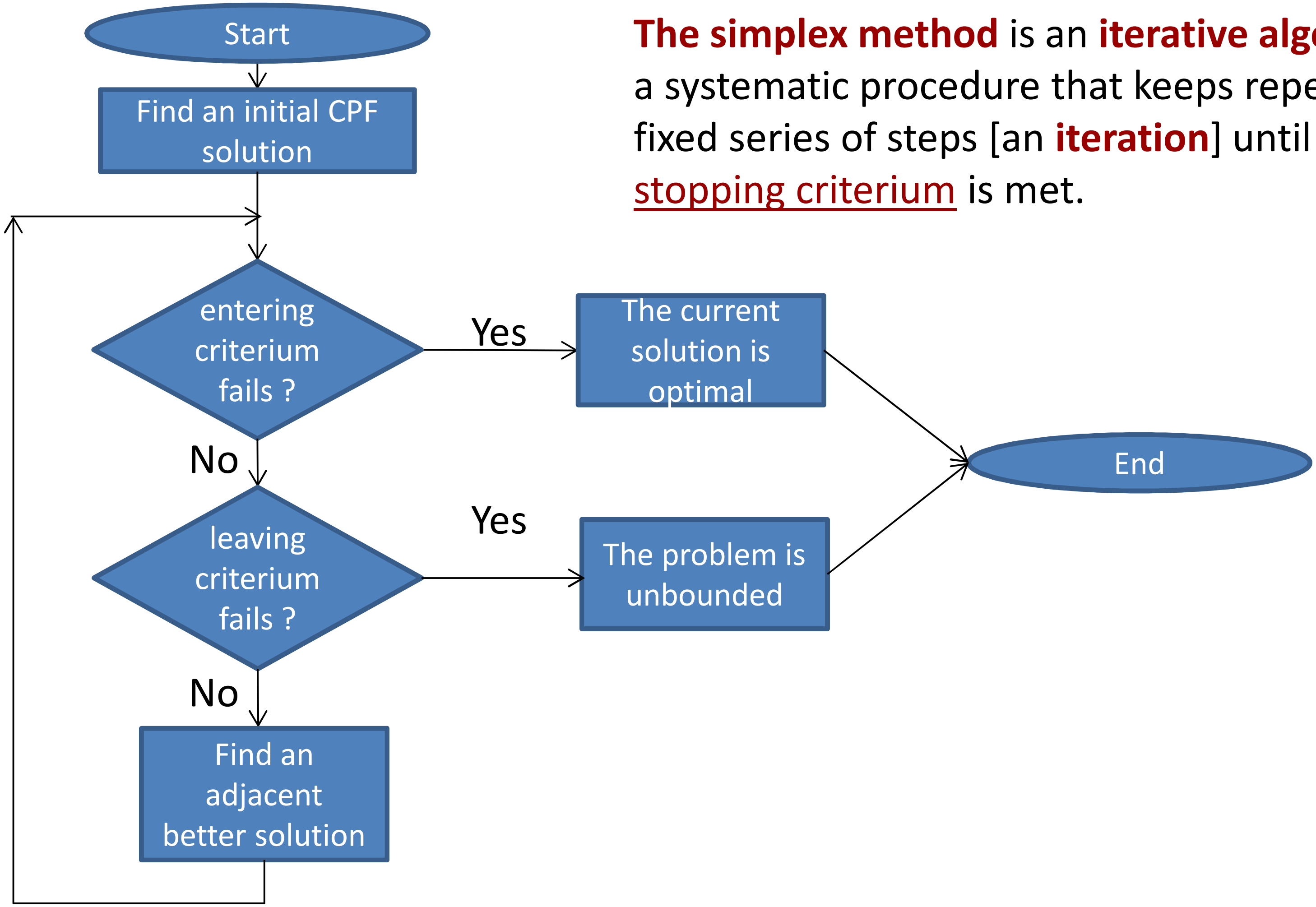
Iteration k

3. If the values in the Objective Function row ≥ 0, STOP (optimal basic solution found).
   Otherwise, EC: $\text{Min}\{\text{coeffic.} < 0$ in the Objective Function row$\}$ [for the new BV x_p]
4. If all the coeffic. in the column of the x_p variable are ≤ 0, STOP (unbounded Obj. Function).
   Otherwise, LC: $\text{Min}\left\{\dfrac{\text{RHS}}{\text{coeffic.}>0 \text{ in the column of } x_p}\right\}$ [for the new NBV x_r]
5. Update the simplex tableau ⇨ new BFS: x_p BV; x_r NBV;

   $k \longleftarrow k+1$; go to 3.

EC=entering criterium;LC=Leaving criterium.

LP – The Simplex Method



The simplex method is an **iterative algorithm** - a systematic procedure that keeps repeating a fixed series of steps [an **iteration**] until a stopping criterium is met.